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Powers of the Vandermonde determinant and the quantum Hall effect

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Abstract. The expansion of the Laughlin ansatz for describing the ground-state wavefunction for the fractional quantum Hall effect as a linear combination of Slater determinantal wavefunctions for N particles is discussed in terms of the corresponding expansion of even powers of the Vandermonde alternant into Schur functions. Two new algorithms for computing the coefficients of the complete expansion are given. They appear to be substantially more efficient than other methods and avoid any use of symmetric group characters. A number of examples are given and the results obtained for $N = 7, 8$ and 9 reviewed. The separate calculation of individual coefficients is also discussed.

1. Introduction

The theory of symmetric functions is quite old, appearing even in Newton's *Arithmetica Universalis.* A modern account is given by Macdonald (1979) which we shall use as reference for matters of notation. Applications of symmetric functions to problems in physics abound. The Schur functions are particularly relevant to discussions of the quantum Hall effect (Stone 1990). Laughlin (1983) has given an ansatz describing the ground-state wavefunction for the fractional quantum Hall effect which involves the fractional filling of the lowest Landau level. There has been considerable recent interest in the expansion of the Laughlin wavefunction as a linear combination of Slater determinantal wavefunctions for N particles (Dunne 1993, Di Francesco *et al* 1993).

It is found that the even powers of the Vandermonde alternating function play a key role in determining the coefficients of the expansion of the Laughlin wavefunction as a linear combination of Slater determinantal wavefunctions. Indeed, the relevant coefficients are directly related to the signed integer coefficients that arise in the expansion of the even powers of the Vandermonde alternating function into Schur functions. The problem of determining the expansion coefficients for increasingly large values of N is combinatorially explosive.

Our principal result is the creation of two algorithms for computing the expansion coefficients in a more efficient manner than in hitherto stated methods and these algorithms entirely avoid the need for any knowledge of symmetric group characters. We give a brief

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statement of the problem and then state the algorithms, illustrating their application by simple examples. Then we review some previous works concerned with the calculation of particular coefficients. Finally we comment on some details of our computed results.

2. The expansion of the Laughlin wavefunction

Laughlin (1983) describes the fractional quantum Hall effect in terms of a (unnormalized) wavefunction

$$\Psi^m_{\text{Laughlin}}(z_1,\ldots,z_N)=\prod_{i<j}^{N}(z_i-z_j)^{2m+1}\exp\left(-\tfrac{1}{2}\sum_{l=1}^{N}|z_l|^2\right) \tag{1}$$

where $z=x+\mathrm{i}y$ and m is an integer corresponding to states of fractional filling $1/(2m+1)$ of the lowest Landau level. The Laughlin wavefunction has a fixed angular momentum

$$J_{\text{Laughlin}}=(2m+1)\tfrac{1}{2}N(N-1) \tag{2}$$

and may be expanded as a linear combination of Slater determinantal wavefunctions having the same angular momentum (Dunne 1993, Di Francesco *et al* 1993).

The Vandermonde alternating function in N variables is defined as

$$V(z_1,\ldots,z_N)=\prod_{i<j}^{N}(z_i-z_j). \tag{3}$$

While V is an alternating function, an even power of V, say V^{2m}, is necessarily a *symmetric* function and, hence, must be expandable in any suitable linear integral basis of symmetric functions such as the Schur functions

$$s_\lambda(z_1,\ldots,z_N)=\{\lambda\}=\{\lambda_1,\ldots,\lambda_p\} \tag{4}$$

which are indexed, in this case, by partitions of the integer

$$n=mN(N-1). \tag{5}$$

Dropping questions of normalization, we may write

$$\frac{\Psi_{\text{Laughlin}}}{V}=V^{2m}=\sum_{\lambda\vdash n}c^\lambda\{\lambda\}. \tag{6}$$

The coefficients c^λ are signed integers and are precisely the same integers that arise in the expansion of the Laughlin wavefunction as a linear combination of Slater determinants. Of particular interest is the determination of the expansion coefficients as the number of particles N increases.

It suffices to calculate the expansion coefficients for $m=1$ as the coefficients for higher values of m can be found from these by simple application of the Littlewood–Richardson rule (LR rule) (cf Macdonald 1979) for the multiplication of Schur functions. The partitions (λ), indexing the Schur functions for $m=1$, are of weight $N(N-1)$, where N is the number of particles (or equivalently the number of variables z_i $i=1,\ldots,N$). It is this feature that makes the problem combinatorially explosive. Di Francesco *et al* (1993) have obtained an expression for the total number of admissible partitions as a function of N. While the evidence prior to our work indicated that the coefficients associated with the admissible partitions are all non-zero, a proof has been lacking and indeed we have found a counter example.

Dunne (1993) has calculated the expansion coefficients for V^2 for up to $N = 6$ variables. His method involves expressing the elements of the Vandermonde determinant in terms of power-sum symmetric functions and then using the character tables of $S_{N(N-1)}$ to transform the power sums into Schur functions. Thus, for eight variables, one already requires knowledge of a significant portion of the character table of S_{56}. Both Dunne and Di Francesco *et al* note that the coefficients display certain symmetries but not enough to significantly reduce the problem for even small values of N. We now turn to the statement of our expansion algorithms.

3. Algorithms for the complete expansion

3.1. The first algorithm

The first algorithm is based upon a simple property of the operator (acting on the space of polynomials $\mathbf{C}[x_1, \ldots, x_n]$) Ω_n which sends a monomial in n variables $z^\lambda = z_n^{\lambda_1} \cdots z_1^{\lambda_n}$ (the usual order being reversed) onto the Schur function $s_\lambda(z_1, \ldots, z_n)$. That is

$$\Omega_n(z^\lambda) = \sum_{w \in S_n} w(z^{\lambda+\rho_n}/V_n) \tag{7}$$

where $\rho_n = (n-1, n-2, \ldots, 1, 0)$ and $V_n = \prod_{1 \leqslant i < j \leqslant n}(z_i - z_j)$. The required property is as follows.

Lemma 3.1. $\Omega_n \Omega_{n-1} = \Omega_n$. This lemma is a direct consequence of the following two facts:

(i) If a polynomial $f(z_1, \ldots, z_n)$ is symmetric in $z_1, \ldots, z_n$ then $\Omega_n(f) = f$. This follows, for example, from the factorized form of Ω_n, as given by Lascoux and Schützenberger (1983) (see also Macdonald (1991) pp 27–8). That is, if π_i denotes the isobaric divided-difference operator $f \longrightarrow (z_i f - z_{i+1}\sigma_i f)/(z_i - z_{i+1})$ (σ_i being the transposition of z_i and z_{i+1}), then, for any reduced decomposition $\omega = \sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_r}$ of the permutation $\omega = (n, n-1, \ldots, 2, 1)$, one has $\Omega_n = \pi_{i_1}\pi_{i_2} \cdots \pi_{i_r}$. Since a polynomial which is symmetric in z_i and z_{i+1} is clearly a scalar for π_i, then a completely symmetric polynomial is also a scalar for Ω_n.

(ii) One has

$$\Omega_n(z_n^k s_\lambda(z_1, \ldots, z_{n-1})) = s_{(k,\lambda)}(z_1, \ldots, z_n). \tag{8}$$

This is a particular case of proposition 8.8 of Lascoux (1990).

This allows for the recursive computation of the Schur expansion of sequences of symmetric functions of the form $F_n = F_{n-1}U_n$, where F_{n-1} depends only on $z_1, \ldots, z_{n-1}$ and U_n is a reasonably small factor involving the n variables which is symmetric in the first $n-1$ ones. For example, with $F_n = V_n^2$, the starting point is as follows. For $n = 2$, $V_2^2 = (z_1 - z_2)^2 = z_1^2 - 2z_1z_2 + z_2^2$ and since it is a symmetric function

$$V_2^2 = \Omega_2(V_2^2) = s_{02} - 2s_{11} + s_{20} = s_2 - 3s_{11}.$$

To pass to $n = 3$, one writes

$$V_2^2 = \Omega_2(z_2^2 - 3z_1z_2)$$

and

$$V_3^2 = \Omega_3\left(\Omega_2(z_2^2 - 3z_1z_2)\cdot(z_1 - z_3)^2(z_2 - z_3)^2\right).$$

The factor $U_3 = (z_1 - z_3)^2(z_2 - z_3)^2$, being symmetric in z_1, z_2, is a scalar for the linear operator Ω_2 and can be incorporated into its argument. We get

$$\begin{aligned} V_3^2 &= \Omega_3(\Omega_2((z_2^2 - 3z_1z_2)(z_1 - z_3)^2(z_2 - z_3)^2)) \\ &= \Omega_3((z_2^2 - 3z_1z_2)(z_1 - z_3)^2(z_2 - z_3)^2) \end{aligned}$$

(by the lemma)

$$\begin{aligned} &= \Omega_3(-3z_1^3z_2^3 + 6z_1^3z_2^2z_3 - 3z_1^3z_2z_3^2 + 4z_1^2z_2^3z_3 - 11z_1^2z_2^2z_3^2 + 6z_1^2z_2z_3^3 + z_1z_2^3z_3^2 \\ &\quad + 4z_1z_2^2z_3^3 - 3z_1z_2z_3^4 + z_1^2z_2^4 - 2z_1z_2^4z_3 + z_2^4z_3^2 - 2z_2^3z_3^3 + z_2^2z_3^4) \\ &= s_{42} - 3s_{33} - 3s_{411} + 6s_{321} - 15s_{222}. \end{aligned}$$

3.2. The second algorithm

The second algorithm is an improvement on the preceding one, making use of the expansion of the coefficients of $U_n = \prod_{i<n}(z_i - z_n)^2$ in terms of Schur functions of $z_1, \ldots, z_{n-1}$. One has to note, however, that the first algorithm does not need any specialized routine for handling symmetric functions and can be easily implemented with any general-purpose computer-algebra system. The second algorithm requires the Littlewood–Richardson rule, or at least Pieri's rule.

By writing, as above,

$$V_n^2 = V_{n-1}^2 U_n \tag{9}$$

the product is then computed by the Littlewood–Richardson rule and to obtain the expansion of V_n^2 in Schur functions, one just has to apply Ω_n to the expansion of (9), taking into account formula (8).

For example, to compute V_3^2, one uses

$$V_2^2 = s_2 - 3s_{11}$$

and

$$(z_1 - z_3)^2(z_2 - z_3)^2 = s_{22} - 2z_3s_{21} + z_3^2(3s_{11} + s_2) - 2z_3^3s_1 + z_3^4. \tag{10}$$

Multiplying these two expressions by the LR rule (restricted to partitions of length $\leqslant 2$) yields

$$V_3^2 = z_3^4s_2 - 3z_3^4s_{11} + z_3^2s_4 + z_3^2s_{31} - 8z_3^2s_{22} - 2z_3^3s_3 + 4z_3^3s_{21} + s_{42} - 3s_{33} - 2z_3s_{41} + 4z_3s_{32}.$$

Applying Ω_3 and standardizing the resulting Schur functions, one finds

$$V_3^2 = s_{42} - 3s_{411} + s_{24} + s_{231} - 8s_{222} - 2s_{33} + 4s_{321} + s_{042} - 3s_{033} - 2s_{141} + 4s_{132}$$
$$= s_{42} - 3s_{411} - 3s_{33} - 15s_{222} + 6s_{321}$$

as required.

The expansion of U_n in Schur functions of the first $n-1$ variables can be obtained directly by means of Cauchy's identity and the formula giving the dimensions of the irreducible representations of $GL(2)$. Indeed, setting $t = -z_n^{-1}$,

$$U_n = \prod_{i=1}^{n-1}(z_i - z_n)^2 = t^{2-2n}\prod_{i=1}^{n-1}(1+tz_i)^2 = t^{2-2n}\sum_{\mu} s_{\mu'}(1,1)s_\mu(z_1,\ldots,z_{n-1})t^{|\mu|}$$

so that, taking into account the fact that a Schur function is zero when the length of the partition exceeds the number of variables,

$$U_n = \sum_{\ell(\mu')\leqslant 2,\ \ell(\mu)\leqslant n-1} s_{\mu'}(1,1)s_\mu(z_1,\ldots,z_{n-1})(-z_n)^{2n-2-|\mu|} \qquad (11)$$

where μ' denotes the conjugate partition of μ. The general formula giving the value of $s_\lambda(1,\ldots,1)$ can be found in Macdonald (1979, ex 4, p 28). In the case where $\mu' = \lambda = (\lambda_1, \lambda_2)$, the determinantal expression of s_λ gives

$$s_\lambda(1,1) = \binom{\lambda_1+1}{2}\binom{\lambda_2+1}{2} - \binom{\lambda_1+2}{2}\binom{\lambda_2}{2}.$$

This version is more complicated but saves a good deal of memory. Indeed, the number of monomials in U_n is 3^{n-1} but the number of non-zero terms when expressed in Schur functions of $z_1,\ldots,z_{n-1}$ is only $n(n+1)/2$.

Another variant of this method uses the fact that $U_n = G_n^2$, where

$$G_n = \prod_{i<n}(z_i - z_n) = \sum_{k=0}^{n-1}(-z_n)^k e_{n-k}(x_1,\ldots,x_{n-1})$$

and performs the multiplication $V_{n-1}^2 U_n$ in two stages

$$V_{n-1}^2 \longrightarrow V_{n-1}^2 G_n \longrightarrow V_{n-1}^2 G_n^2$$

the products $s_\mu e_r$ being expanded by means of Pieri's formula (see Macdonald 1979, formula (5.17) p 42).

Combinatorial identities involving sums of Schur functions indexed by shapes contained in a rectangle can be used to derive properties of the coefficients in equation (11). For example, a generating function which was conjectured by MacMahon and proved by Andrews (see Macdonald 1979, formula (4) p 52) shows (by putting $q = -1$) that the sum of these coefficients is zero for even values of n and $(n+1)/2$ for odd values of n. Also, the same identity (with $q = 1$) proves that the sum of their absolute values is $n(n+1)(n+2)/6$.

Both algorithms also work for the powers of the discriminant (with a suitable modification for the second algorithm, that is, the expansion of U_n^k will use dimensions for $GL(2k)$ or $2k$ applications of Pieri's formula). However, it is perhaps faster to compute higher powers from the first algorithm by use of the LR rule. The main inconvenience of these methods is that they can give only the complete expansion. To investigate the individual coefficients, one probably has to look at the q-analogue. In this case, the coefficients have a nice expression, although they are not amenable to practical computations. However, the two methods explained above can also give the complete expansion of the q-discriminant.

4. The q-discriminant

Consider the sequence of polynomials

$$R_n(q) = \prod_{1 \leqslant i,j \leqslant n} (z_i - qz_j). \tag{12}$$

One has $R_n(q) = (1-q)^n(z_1 \cdots z_n)D_n(q)$, where $D_n(1) = (-1)^{n(n-1)/2}V_n^2$. The Schur expansion of the polynomial $R_n(-q)$ gives the graded decomposition of the exterior powers of the adjoint representation of $GL(n)$ and has been investigated by several authors (Stanley 1984, Hanlon 1985, Stembridge 1987, Thibon 1990, Berenstein and Zelevinsky 1992, Kirillov 1992). In particular, the coefficient of the Schur function $s_{(n^n)}$ in $R_n(-q)$ is the Poincaré polynomial of the unitary group $U(n)$. It is a classical result of Weyl (1939) that this polynomial is equal to

$$P_n(q) = (1+q)(1+q^3)\cdots(1+q^{2n-1})$$

(see Littlewood (1953) for a simple proof using symmetric functions) so that the coefficient of $s_{(n-1)^n}$ in V_n^2 is equal to $(-1)^{n(n-1)/2}(2n-1)!!$

For two finite sets of variables A and B, define the *resultant* of A and B as

$$R(A, B) = \prod_{i,j}(a_i - b_j). \tag{13}$$

If $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_m\}$ then a formula by Cauchy (see e.g. Lascoux (1990) for this formulation) states that the resultant can be expressed as a super Schur function

$$R(A, B) = s_{(m^n)}(A - B). \tag{14}$$

With $A = Z = \{z_1, \ldots, z_n\}$ and $B = qZ$, this gives, in λ-ring notation

$$R_n(q) = s_{(n^n)}((1-q)Z) = \sum_\lambda s_{(n^n)} * s_\lambda(1-q)s_\lambda(Z) \tag{15}$$

where $*$ denotes the internal product of symmetric functions (corresponding to the Kronecker product of S_n representations). That is, the decomposition coefficients are compactly expressed as q-specializations of Kronecker products (we recall that symmetric functions of the argument $1-q$ can be defined by replacing each power sum p_k by the polynomial $1-q^k$). That is

$$(s_{(n^n)} * s_\lambda)(1-q) = \sum_\alpha \chi_\alpha^{(n^n)} \chi_\alpha^\lambda \prod_i (1-q^{\alpha_i}).$$

Since $s_\mu(1-q) \neq 0$ only when μ is a hook, the symmetry of the CG numbers for S_n shows that the problem is equivalent to finding the Kronecker product of a square character by a hook character.

For those partitions λ of n^2 which have the special form

$$\lambda = ((n-1)^n) + \mu = (\mu_1 + n - 1, \mu_2 + n - 1, \ldots, \mu_n + n - 1) \tag{16}$$

where μ is a partition of n, an exact formula has been given by Stembridge (1987). In λ-ring notation, the result can be stated as

$$s_{(n^n)} * s_{(n-1)^n+\mu}(1-q) = (-1)^n (q^2;q^2)_n \, s_\mu\left(\frac{-1}{1+q}\right) \tag{17}$$

where, as usual, $(q^2;q^2)_n = (1-q^2)(1-q^4)\cdots(1-q^{2n})$.

Using the formula (see e.g. Macdonald 1979)

$$s_\mu\left(\frac{a-b}{1-q}\right) = q^{n(\mu)} \prod_{x\in\mu} \frac{a-bq^{c(x)}}{1-q^{h(x)}} \tag{18}$$

one recovers the more explicit form given in Stembridge (1987) leading to the coefficient g_n^ν for s_ν with $\nu = (n-2)^n + \mu$ in the expansion of V_n^2 being given by

$$g_n^\nu = (-1)^{n(n+1)/2} \frac{(q^2;q^2)_n}{(1-q)^n} \prod_{(i,j)\in\mu} \frac{q^{2i-1}-q^{2j-2}}{1-q^{2h(i,j)}}\Bigg|_{q\to 1}. \tag{19}$$

For example, with $n = 8$, the predicted coefficients are (for the discriminant)

$$\begin{gathered} -135\,135,\ 218\,295,\ -56\,700,\ -297\,675,\ -8820,\ 100\,800,\ 385\,875,\\ -1890,\ 15\,750,\ -37\,800,\ -141\,750,\ -496\,125,\ -5670,\ -17\,640,\ 66\,150,\\ 181\,440,\ 654\,885,\ -22\,050,\ -79\,380,\ -207\,900,\ -945\,945,\ 2\,027\,025 \end{gathered}$$

corresponding to the partitions μ

$$\begin{gathered} (8),\ (7,1),\ (6,2),\ (6,1,1),\ (5,3),\ (5,2,1),\ (5,1,1,1),\ (4,4),\ (4,3,1),\\ (4,2,2),\ (4,2,1,1),\ (4,1,1,1,1),\ (3,3,2),\ (3,3,1,1),\ (3,2,2,1),\\ (3,2,1,1,1),\ (3,1,1,1,1,1),\ (2,2,2,2),\ (2,2,2,1,1),\ (2,2,1,1,1,1),\\ (2,1,1,1,1,1,1),\ (1,1,1,1,1,1,1,1) \end{gathered}$$

which have to be shifted by $(6,6,6,6,6,6,6,6)$, e.g. the coefficient of $s_{127\,766\,666}$ in V_8^2 is equal to $-297\,675$.

Equation (19) can lead to useful general expressions for certain types of partitions. Thus for partitions where $\mu = (n-k-1, 1^k)$, one obtains

$$g_n^\lambda = (-1)^{n(n+1)/2}(-1)^{k+1}\binom{n-1}{k}(2k+1)!!(2n-2k-3)!! \tag{20}$$

More generally, the coefficient of s_λ in V_n^{2m} is equal to $(-1)^{mn(n-1)/2} c_\lambda^{(m)}(1)$, where

$$c_\lambda^{(m)}(q) = (1-q)^{-mn}(s_{(mn)^n} * s_\lambda)(1-mq).$$

When $\lambda = (mn-m-1)^n + \mu$, with μ a partition of n, then theorem 6.1. of Stembridge (1987) shows that

$$(s_{(mn)^n} * s_\lambda)(1-mq) = \frac{(q;q)_{n(m+1)}}{(q;q^{m+1})_n (q;q^m)_n} s_{\mu'}\left(\frac{1-q}{1-q^{m+1}}\right). \tag{21}$$

Again, the symmetric functions of the argument $(1-mq)$ are defined by replacing each power sum p_k by the polynomial $(1-mq^k)$.

5. Symmetries of coefficients

Dunne (1993) has noted that the coefficients g_n^λ, associated with the expansion expansion of V_n^2, exhibit certain symmetries, most notably he finds that the coefficients associated with a partition λ and what he terms the *reversed* partition

$$\{\bar{\lambda}\} = \{2(n-1) - \lambda_n, \ldots, 2(n-1) - \lambda_1\} \tag{22}$$

are identical. An equivalent statement can be made by noting that V_n^2 may equally as well be expanded in terms of characters of the special unitary group SU_n and noting that the coefficients of contragredient representations are identical. This leads to an additional check on the calculation of the coefficients for the complete expansion of V_n^2 since both the sum of the dimensions and of the second-order Dynkin index must come to zero; the latter giving a more stringent test than the former. The set of S-functions associated with a given multiplicity must be reversal invariant or, in the case of SU_n, self-contragredient.

6. Recursion relations

Taking into account the fact that

$$U_{n+1} = x_{n+1}^{2n} + \cdots + (x_1 x_2 \cdots x_n)^2$$

it follows from (9) that

$$g_{n+1}^{2n,\lambda} = g_n^\lambda \tag{23a}$$

and a similar argument, using the variable z_1 for the recursion instead of z_{n+1}, with the corresponding symmetrizing operator implies as well that

$$g_{n+1}^{\lambda+2^n} = g_n^\lambda. \tag{23b}$$

If the λ are given in reverse lexicographic order then equation (23a) gives all the coefficients of $V_{(n+1)^2}$, whose leading part is $2n$, directly from those found for V_n^2.

7. Summary of results

Both algorithms given in section 3 have been independently applied in the computing packages SYMMETRICA† and SCHUR‡ to compute the expansion coefficients for up to, and including, $N = 8$ (and the second one served for computing $N = 9$). For the case of $N = 8$, the largest coefficient is 2 027 025 and the absolute sum of the complete set of expansion coefficients is 41 603 200. It is interesting to note that for $N = 7$ there are 1111 distinct partitions in agreement with the number of admissible tableaux calculated by Di Francesco *et al* (1993), whereas for $N = 8$ there are 5294 distinct partitions, eight fewer than the

† SYMMETRICA is a computing package for calculating with symmetric functions and symmetric groups. For further information e-mail: axel@btm2x2.mat.uni-bayreuth.de

‡ SCHUR is an interactive program for calculating the properties of Lie groups and symmetric functions by Brian G Wybourne, distributed by S Christensen, PO Box 16175, Chapel Hill, NC 27516 USA. e-mail: stevec@wri.com

calculated number of admissible tableaux showing that eight of the admissible tableaux are associated with zero coefficients. These partitions are

$$(13, 11, 9, 8, 5, 5, 4, 1), (13, 11, 9, 8, 5, 4, 4, 2), (13, 11, 9, 7, 6, 5, 4, 1),$$
$$(13, 10, 9, 9, 6, 5, 3, 1), (13, 10, 9, 8, 7, 5, 3, 1), (12, 11, 9, 7, 7, 4, 4, 2),$$
$$(12, 10, 10, 9, 6, 5, 3, 1), (12, 10, 10, 7, 7, 5, 3, 2).$$

For $N = 9$, the number of admissible partitions is 26376, but there are only 26310 distinct partitions involved.

The results for $N = 7, 8, 9$ are voluminous. Copies of the computer output are available as a TEXfile distributed via e-mail (bgw@phys.uni.torun.pl).

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